PHYSICAL REVIEW E VOLUME 59, NUMBER 1 JANUARY 1999

Influence of a periodic field on two-level classical systems

V. Berdichevsky and M. Gitterman
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
(Received 18 September 1998)

An external periodic field is able to change the asymptotic values of the populations of two asymmetric energy levels in a bistable potential tending to equalize the populations of the discrete levels, or even to reverse the populations for space-extended systems. The population of the oscillating well can either decrease or increase compared with the field-free case, depending on the frequency of the external field.
[S1063-651X(99)50201-8]

PACS number(s): 05.40.−a, 05.60.−k, 05.70.Ln

The investigation of nonlinear dynamical systems subject to a periodic and/or random field has attracted considerable interest. Recent reviews describe different, quite complicated models and their applications to many physical, chemical, and biological systems [1–3]. In this Rapid Communication we consider discrete and space-extended two-level asymmetric systems where, in contrast to the usual approach, the probabilities of the transitions $1\rightarrow 2$ and $2\rightarrow 1$ are different, i.e., in the absence of an external field, the two levels have nonequal populations as $t\rightarrow\infty$.

Although the effect of potential asymmetry on stochastic resonance has been considered earlier [2], the problem of the changes in populations in the presence of an external field remain unsolved, and this is the subject of our study. It turns out that, in addition to periodic changes of the populations, an external periodic field tends to equalize the populations of the two levels as $t\rightarrow\infty$, stabilizing the lower ''metastable'' level or even reversing the populations of these levels. Moreover, the populations are nonmonotonic functions of the frequency of the external field.

The simplest model of a system that can be found either in the ''left'' (population n_1), or in the ''right'' (population n_2) states, where $n_1+n_2=1$, has many applications in science, and has been considered repeatedly in the literature. Our approach is close to that of Refs. [5,6] with the essential difference lying in the assumption that, say, the left state is less stable, i.e., the potential barrier U_1 for transmission to the right state is lower than U_2 for the reverse transition, $U_1<U_2$.

The rate equation describing the dynamics of a two-state system has the following form:

$$\frac{dn_1}{dt}=-\frac{dn_2}{dt}=W_2n_2-W_1n_1=W_2-(W_1+W_2)n_1, \quad (1)$$

where $W_{1(2)}$ is the transition rate out of state 1(2), and is assumed to have the simple Arrhenius form.

In the absence of an external field, one finds for $t\rightarrow\infty$, when the initial conditions are washed out

$$n_{1,\infty}=\frac{W_2^o}{W_1^o+W_2^o};\quad n_{2,\infty}=\frac{W_1^o}{W_1^o+W_2^o};$$
$$W_1^0\equiv e^{-(U_1/D)};\quad W_2^0\equiv e^{-(U_2/D)} \quad (2)$$

i.e., $n_{1,\infty}<n_{2,\infty}$ — the left ''shallow'' state contains fewer particles than the right ''deep'' state. In Eq. (2) we neglect the possible change of prefactors in exponentials due to asymmetry.

The influence of the external periodic field is usually described by the modulation of the potential well, i.e., U_1 is replaced by $U_1+A\cos(\Omega t)$, and U_2, by $U_2-A\cos(\Omega t)$. We accept this assumption hereafter.

Substituting the modulated barrier heights inside the rate equation (1), one concludes that, after a transient period, the solution of this equation becomes periodic in time:

$$n_1=n_{1,\infty}+\sum_m [A_m\cos(m\Omega t)+B_m\sin(m\Omega t)]$$
$$=n_{1,\infty}+\sum_m \sqrt{A_m^2+B_m^2}\sin(m\Omega t+\phi_m). \quad (3)$$

Substituting Eq. (3) and the expansion of $\exp(\pm[A\cos(\Omega t)]/D)$ in a series of modified Bessel functions of the first kind [4] into Eq. (1), one can find recursive relations for $n_{1,\infty}$, A_m, and B_m, as has been done for similar problems [5,6]. Truncating the recursive relations at $m=0,1,2\ldots$, one obtains a set of coefficients in Eq. (3) that corresponds to increasing powers of (A/D), i.e., of the amplitude of the external field. Omitting the straightforward calculations, one finds the following results to the lowest order in the field amplitude, i.e., to first order for A_1 and B_1, and to second order for $n_{1,\infty}$:

$$n_{1,\infty}=\frac{W_2^0}{W_1^0+W_2^0}+\frac{A^2}{D^2}\frac{W_1^0W_2^0}{\Omega^2+(W_1^0+W_2^0)^2}\frac{W_1^0-W_2^0}{W_1^0+W_2^0},$$
$$\sqrt{A_1^2+B_1^2}=\frac{2AW_1^oW_2^o}{D(W_1^o+W_2^o)[\Omega^2+(W_1^o+W_2^o)^2]^{1/2}}. \quad (4)$$

In all previous analyses ([5,6] and others), two stable states ($W_1^0=W_2^0$) have been considered and the limiting ($t\rightarrow\infty$) values of $n_{1,\infty},n_{2,\infty}$ did not change in the presence of an external periodic field. The only influence of this field was to produce a periodic change of the population of the two states described by the coefficients A_m and B_m in Eq. (3).

As one can see from Eq. (4), in the presence of a field, the field-free expression for $n_{1,\infty}$ is augmented by an additional

FIG. 1. Square double-well potential with an oscillating left well.

positive term (since $W_1^0 > W_2^0$). One can show that more positive terms will come from the next-order corrections in (A/D). Thus, one finds that the less stable state becomes ''more stable'' in the presence of an external periodic field. In fact, this field not only equalizes the populations of the two states; under some circumstances (see below), it can even reverse them.

The second conclusion, which follows from Eq. (4), is that the amplitude of the oscillations $(A_1^2+B_1^2)^{1/2}$ is monotonic as a function of the external field frequency Ω, but nonmonotonic as a function of the noise strength D (stochastic resonance [2]).

We now turn to space-extended systems and, as an example, consider a particle moving in the piecewise double-well potential $U(x)$ under the influence of white noise. In addition, we assume that the left well is subject to an external periodic field, as shown in Fig. 1, where the length is expressed in arbitrary units.

The Fokker-Planck equation for the probability function $P(x,t)$ for the position x of a diffusive particle at the time t is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} P + D\frac{\partial P}{\partial x}\right) \equiv -\frac{\partial J}{\partial x}, \quad (5)$$

where the probability current J is defined in Eq. (5).

For the potentials $U(x)$ shown in Fig. 1, everywere except at matching points, $\partial U/\partial x = 0$ and Eq. (5) reduces to a simple diffusion equation. Moreover, our choice for the periodic signal does not introduce an additional force in Eq. (5), which still maintains the form of a simple diffusion equation. However, the periodic signal enters the matching conditions, namely, one has to solve Eq. (5) in each region of $U(x) =$ const, and then ensure the continuity of P and J on the boundaries of these regions. The matching conditions have to be complemented by reflected boundary conditions at the walls.

Our main assumption is the smallness of the amplitude of the external field, which means $(A/D) < 1$, and accordingly we seek the solution of Eq. (5) in each region m as

$$P_m = S_m + \sum_{l=1}^{\infty}\left(\frac{A}{D}\right)^l f_m^{(l)}(x,t), \quad (6)$$

where $f_m^{(l)}$ is a periodic function of t, which can be written in the following form:

$$f_m^{(l)} = f_{m,0}^{(l)} + \sum_{k=1}^{\infty}\,[(f_{m,k}^{(l)} e^{r_k x} + \tilde{f}_{m,k}^{(l)} e^{-r_k x}) e^{i\Omega k t} + \text{c.c.}], \quad (7)$$

$$r_k \equiv \sqrt{\frac{i\Omega k}{D}}.$$

We keep only the lowest nonvanishing corrections to the field-free asymptotic probabilities S_m in the amplitude of the external field. Due to the boundary conditions, to the first order in the field, the time-independent corrections vanish, and so we also keep the nonoscillating second-order terms in (A/D). An expansion of $\exp(\pm[A\cos(\Omega t)]/D)$ up to first order in (A/D) will influence only the $k=1$ terms in Eq. (7), which take the following form

$$P_m = S_m + \frac{A}{D}[(f_m e^{rx} + \tilde{f}_m e^{-rx}) e^{i\Omega t} + \text{c.c.}] + \left(\frac{A}{D}\right)^2 g_m. \quad (8)$$

Equation (8) contains twelve parameters $(S_m, f_m, \tilde{f}_m, g_m;\ m=1,2,3)$ that satisfy twelve equations [two on the wells and two on each of two boundaries for each order in the small parameter (A/D)]. We will present the full solution elsewhere. Here we restrict our consideration to the time-independent populations [up to the second order in $(A/D)^2$] of the right, $n_{r,\infty}$, and the left (oscillating), $n_{l,\infty}$, wells:

$$n_{l,\infty} = \int_{-1}^{0} P_1(x,t)\,dx = S_1 + \left(\frac{A}{D}\right)^2 g_1 + \cdots, \quad (9)$$

$$n_{r,\infty} = \int_{1}^{2} P_3(x,t)\,dx = S_3 + \left(\frac{A}{D}\right)^2 g_3 + \cdots, \quad (10)$$

where

$$S_1 = \frac{e^{U_1/D}}{1+e^{U_1/D}+e^{U_2/D}};\quad S_3 = \frac{e^{U_2/D}}{1+e^{U_1/D}+e^{U_2/D}},$$

$$g_1 = -(1+e^{U_2/D});\ g_3 = -(S_2+S_3)\left[\frac{S_1}{4} + \text{Re}(f_1+\overline{f_1})\right],$$

$$\text{Re}(f_1+\tilde{f}_1) = -\frac{1+e^{-U_2/D}}{2e^{(U_1+U_2)/D}}\,\frac{H + \dfrac{\sin^2(a_2)\sinh^2(a_2)}{4H}}{H - \dfrac{\sinh^2(a_2)-\sin^2(a_2)}{4}}, \quad (11)$$

$$H \equiv \frac{(1+e^{U_1/D}+e^{U_2/D})[\sinh^2(a_1)+\cos^2(a_1)]^2}{e^{U_1/D}+e^{U_2/D}} + \frac{\sinh^2(a_2)-\sin^2(a_2)}{4};\quad a_n \equiv \sqrt{\frac{n^2\Omega}{2D}}.$$

Notice that for the dimensionless length, Ω and D have the same dimensions.

If we assume, as shown in Fig. 1, that without an external field the left well has a lower barrier than the right well, $U_1 < U_2$, one can reverse the population with the help of an external periodic field. The latter will occur when $n_{l,\infty} > n_{r,\infty}$, which can be rewritten as

$$S_3 - S_1 < \left(\frac{A}{D}\right)^2 (g_1 - g_3). \qquad (12)$$

For the latter equation, we can take the values of g_1, g_3, S_1 and S_3 from (11) which gives

$$\left(\frac{A}{D}\right)^2 \frac{1+2e^{U_2/D}}{e^{U_2/D} - e^{U_1/D}} \left[-\mathrm{Re}(f_1 + \overline{f_1}) - \frac{S_1}{4} \right] > 1. \qquad (13)$$

The last inequality is obeyed when the factor in front of the brackets is large, and the expression in the brackets is positive. The former occurs when $A > \sqrt{U_2 - U_1}$, and the latter holds for not-too-small frequencies. Notice that even if it is presumed that $U_{1,2}/D < 1$, the relation between A and $U_2 - U_1$ may be arbitrary.

Hence, we have obtained an interesting result: an external periodic force acting on the ''shallow'' well is able to transform it into the ''deep'' well. This result has been obtained analytically for a small periodic force and, consequently, only for very close minima. However, one can expect that a stronger periodic signal would be able to reverse more distant minima.

Another point of interest is the frequency dependence of second-order in the field correction term [proportional to g_1— see Eq. (9)], which turns out to be nonmonotonic (Fig. 2). Such nonmonotonic dependence of the population on the frequency Ω of an external field has some resemblance to the appropriate quantum effect. Indeed, the quantum tunneling is either enhanced or decreased (''localization'') depending on the frequency of an external field [3].

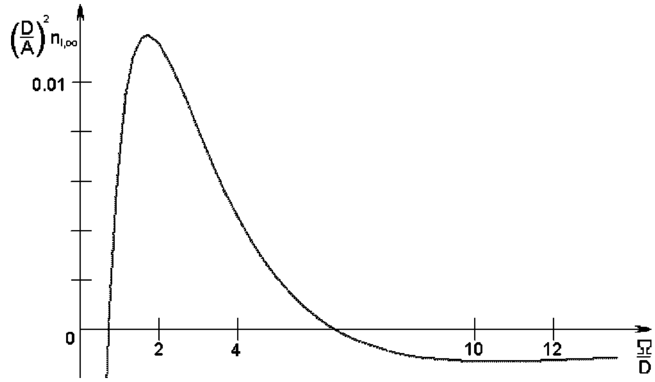

FIG. 2. Change of the left well population [divided by $(A/D)^2$] as a function of the dimensionless frequency Ω/D, of an external field given by Eq. (9), for $U_1/D = 1.0$ and $U_2/D = 1.3$.

In conclusion, we found that a periodic signal tends to equalize the asymptotic populations of two levels, or even to transform a shallow level into a deep level. Moreover, the populations turn out to be nonmonotonic functions of the frequency of an external field so that fields of different frequencies can enhance or decrease transitions between two levels.

The phenomenon described above is somewhat similar to the ''Kapitza pendulum'' whose point of support oscillates vertically, which stabilizes the vertically upward position [7].

[1] P. Hanggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[2] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[3] M. Grifoni and P. Hanggi, Phys. Rep. **304**, 229 (1998).
[4] *Handbook of Mathematical Functions*, 9th ed., edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1970).
[5] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[6] M. Gitterman and J. Kiefer, Physica A **200**, 256 (1993).
[7] L. Landau and E. Lifshitz, *Mechanics*, 2d ed. (Pergamon, Oxford, 1969), p. 95.